

# Growth factor in $f(T)$ gravity

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**ABSTRACT:** We derive the evolution equation of growth factor for the matter over-dense perturbation in  $f(T)$  gravity. For instance, we investigate its behavior in power law model at small redshift and compare it to the prediction of  $\Lambda$ CDM and dark energy with the same equation of state in the framework of Einstein general relativity. We find that the perturbation in  $f(T)$  gravity grows slower than that in Einstein general relativity if  $\partial f/\partial T > 0$  due to the effectively weakened gravity.

**KEYWORDS:** dark energy theory, cosmological perturbation theory.

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## 1. Introduction

The cause for the late-time accelerated expansion of the universe remains one of the most compelling problems in modern physics. Many schemes have been proposed to explain this phenomenon. Although dark energy scenario [1] is the most popular one among them, ones have considered some models based on infra-red modifications to general relativity (GR), such as scalar-tensor theories [2],  $f(R)$  gravity [3] and braneworld models [4]. In general, the resulting field equations are fourth order because the Ricci scalar is constructed from the second order derivatives of the metric, and this feature may lead to pathologies. Recently, an alternative model based on modified teleparallel gravity receives considerable attention. See [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] in detail. Instead of describing gravitational interaction with curvature of the background spacetime by employing the torsionless Levi-Civita connection, one can explore the opposite way and resort to the Weitzenböck connection that has no curvature, in this case torsion will independently do the job that curvature does in GR [20]. There are some terms in the modified Friedmann equation that can be identified as the effective dark energy to give rise to the accelerated expansion of the late-time universe. This paradigm boasts the significant advantage that the field equations are second order, as we shall see in the next section.

In order to discriminate different models, we need to break the degeneracy of background expansion history, and work in first order perturbation to find more information concerning different models [21, 22, 23, 24]. The matter density perturbation which characterizes the inhomogeneities of the universe comes to the rescue. While different models may have the same background behavior, their linear growths of matter perturbation can be quite different. See, for example, [25]. In this paper, we derive the evolution equation

of the linear matter density perturbation and find that it takes the same form as the counterpart in GR, except that the effective Newton's constant is rescaled by a term related to the first derivative of  $f(T)$ . Note that when the Lagrangian of gravitational part is torsion scalar  $T$ , it is equivalent to the Einstein-Hilbert Lagrangian of GR up to a divergence, and hence all behaviors of this theory reduce to those in GR, including the local Lorentz symmetry. While for more general  $f(T)$  gravity, local Lorentz transformation fails as a symmetry of this theory [26], and it is expected that this equation may also provide us some hint about Lorentz violation.

The outline of this paper is as follows. In Sec. 2, we briefly review the theoretical structure of teleparallel gravity and how it explains cosmic acceleration. In Sec. 3, we present the first order equations based on metric perturbations and vierbein perturbations respectively. Moreover, we derive the governing equation for matter density perturbation, then solve it numerically and compare our result to the counterpart of GR. Finally in Sec. 4, we discuss and summarize our results.

## 2. A brief review of $f(T)$ cosmology

In Riemann-Cartan spacetime, the curvature tensor and the torsion tensor coexist. On a manifold, one can define a large number of connections, which differ from each other up to a tensor quantity. The assumptions of torsion-free and metric compatibility lead to the Levi-Civita connection, and this is the one on which Einstein general relativity is based. However, we are free to choose other connections, for instance the Weitzenböck connection which is defined by

$$\Gamma_{\mu\nu}^{\lambda} = e^{\lambda}_A \partial_{\nu} e_{\mu}^A, \quad (2.1)$$

where  $\mathbf{e}_A(x^{\mu})$  is a set of orthonormal vectors, which form a noncoordinate basis for the tangent space at each point on this manifold, and  $\mathbf{e}^A(x^{\mu})$  is the dual vectors. The torsion tensor is given by

$$T^{\lambda}_{\mu\nu} = \Gamma_{\nu\mu}^{\lambda} - \Gamma_{\mu\nu}^{\lambda} = e^{\lambda}_A (\partial_{\mu} e_{\nu}^A - \partial_{\nu} e_{\mu}^A), \quad (2.2)$$

one can find that the curvature tensor and the covariant derivatives of  $\mathbf{e}_A(x^{\mu})$  with respect to this connection vanish globally, therefore  $\mathbf{e}_A(x^{\mu})$  are absolutely parallel vector fields, and this theory is dubbed teleparallel gravity [20]. In this formalism, the fundamental dynamical object is the vierbein field  $e_{\mu}^A(x)$ , and the metric tensor is obtained by a byproduct

$$g_{\mu\nu}(x) = \eta_{AB} e_{\mu}^A(x) e_{\nu}^B(x), \quad (2.3)$$

and then the Levi-Civita connection can be defined in a natural way. The difference between these two connections is described by the contorsion tensor which takes the form

$$K^{\mu\nu}{}_{\rho} = -\frac{1}{2}(T^{\mu\nu}{}_{\rho} - T^{\nu\mu}{}_{\rho} - T_{\rho}{}^{\mu\nu}). \quad (2.4)$$

Finally, we can define a torsion scalar as follows

$$T = S_{\rho}{}^{\mu\nu} T^{\rho}{}_{\mu\nu}, \quad (2.5)$$

where

$$S_{\rho}{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_{\rho} + \delta_{\rho}^{\mu}T^{\alpha\nu}{}_{\alpha} - \delta_{\rho}^{\nu}T^{\alpha\mu}{}_{\alpha}). \quad (2.6)$$

$T$  is the simplest teleparallel Lagrangian, which differs from Einstein-Hilbert Lagrangian only up to a boundary term [27].

Similar to  $f(R)$  gravity, we can write down the Lagrangian for the gravity as a function of  $T$ . The full action reads

$$I = \frac{1}{2\kappa^2} \int d^4x e \cdot [T + f(T)] + \int d^4x e \cdot \mathcal{L}_m, \quad (2.7)$$

where  $e = \det(e_{\mu}{}^A) = \sqrt{-g}$ ,  $\kappa^2 = 8\pi G$  ( $G$  is the Newton's coupling constant), and  $\mathcal{L}_m$  stands for the matter Lagrangian. Performing variation in this action with respect to the vierbein yields the equations of motion

$$\begin{aligned} -\frac{1}{4}e^{\alpha}{}_A[T + f] + e^{\beta}{}_AT^{\mu}{}_{\nu\beta}S_{\mu}{}^{\nu\alpha}[1 + f_T] + e^{-1}\partial_{\mu}(ee^{\rho}{}_AS_{\rho}{}^{\mu\alpha})[1 + f_T] \\ + e^{\rho}{}_AS_{\rho}{}^{\mu\alpha}f_{TT}\partial_{\mu}T = \frac{\kappa^2}{2}e^{\rho}{}_A\mathbf{T}_{\rho}{}^{\alpha}, \end{aligned} \quad (2.8)$$

where the subscript ' $T$ ' denotes the derivative with respect to the torsion scalar and  $\mathbf{T}_{\rho}{}^{\alpha}$  is the energy-momentum tensor. Comparing to the equation in  $f(R)$  gravity which is fourth order differential equation, this equation has an advantage of being second order. So it is much easier for us to analyze it.

From now on, we focus on the a spatially flat Friedmann-Robtson-Walker (FRW) universe only filled with dust-like matter. The metric is given by

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2. \quad (2.9)$$

In this case the torsion scalar is related to the Hubble parameter  $H \equiv d \ln a / dt$  by

$$T = -6H^2. \quad (2.10)$$

The background equations of motion become [6]

$$H^2 = \frac{\kappa^2}{3}\rho - \frac{f}{6} - 2H^2f_T, \quad (2.11)$$

$$\dot{H} = -\frac{1}{4} \frac{6H^2 + f + 12H^2f_T}{1 + f_T - 12H^2f_{TT}}, \quad (2.12)$$

where  $\rho$  is matter energy density. Evidently, the last two terms at the right-hand side of Eq.(2.11) can be explained as the effective dark energy whose energy density is given by

$$\rho_{de} = \frac{1}{2\kappa^2}(-f + 2Tf_T), \quad (2.13)$$

and the corresponding equation of state is

$$w = -1 + \frac{(f - T - 2Tf_T)(f_T + 2Tf_{TT})}{(1 + f_T + 2Tf_{TT})(f - 2Tf_T)}. \quad (2.14)$$

From the above equation, the effective dark energy becomes an effective cosmological constant if

$$f - T - 2Tf_T = 0, \quad (2.15)$$

or

$$f_T + 2Tf_{TT} = 0, \quad (2.16)$$

for all  $T$ . The solution of Eq.(2.15) is

$$f(T) = -T + c_1\sqrt{-T}. \quad (2.17)$$

But now  $(1 + f_T + 2Tf_{TT})$  equals to zero as well. So this solution does not provide an effective cosmological constant. Switch to Eq.(2.16), the solution is

$$f(T) = c_2\sqrt{-T} - \kappa^2\Lambda, \quad (2.18)$$

where  $\Lambda$  is a constant. Substituting the above solution into Eq.(2.13), we find that the effective dark energy density is simplified to be  $\Lambda$  which is nothing but a cosmological constant and the term  $c_2\sqrt{-T}$  does not contribute to the effective energy density at all. To summarize,  $f(T)$  can be taken as a cosmological constant only when  $f(T)$  is a constant. However it is quite trivial.

### 3. Growth factor

The early universe was made very nearly uniform by an inflationary state. The origin of structure in the universe is seeded by the small quantum fluctuations generated at the inflationary epoch. These small perturbations over time grew to become all of the structure we observe. Once the universe becomes matter dominated primeval density inhomogeneities ( $\delta\rho/\rho \sim 10^{-5}$ ) are amplified by gravity and grow into the structure we see today. In this section, we investigate how the matter density perturbation grows in  $f(T)$  gravity. We keep terms up to the first order in the perturbed vierbein field. For the sake of simplicity, we will work in Newtonian gauge, which is valid for  $f(T)$  gravity theory because it still preserves the principle of general covariance.

This section is divided into three subsections. Firstly, we follow the approaches in [18] and define all the scalar degrees of freedom in the perturbed metric in Newtonian gauge. We find that this ansatz is too naive and problematic. In the second subsection, we start with a general perturbed vierbein field which includes more degrees of freedom and derive the correct evolution equation for the matter density perturbation. In the last subsection, we consider a concrete  $f(T)$  model and compare the growth of matter over-dense perturbation in  $f(T)$  gravity with that in GR.

#### 3.1 A naive ansatz for the perturbed vierbein

Up to the linear order, scalar perturbations should decouple with vector and tensor perturbations. The perturbed FRW metric can be written by

$$ds^2 = (1 + 2\phi)dt^2 - a^2(t)(1 - 2\psi)\delta_{ij}dx^i dx^j. \quad (3.1)$$

A naive ansatz for the perturbed vierbein can be written by

$$e_\mu^A = \begin{pmatrix} 1 + \phi & 0 \\ 0 & a(1 - \psi)\delta_i^m \end{pmatrix}. \quad (3.2)$$

Accordingly, the perturbed energy-momentum tensor takes the form

$$\delta \mathbf{T}_\mu^\nu = \begin{pmatrix} -\delta\rho & -a^{-2}(\rho + p)\partial_i v \\ (\rho + p)\partial_i v & \delta_i^j \delta p \end{pmatrix}, \quad (3.3)$$

where  $\rho$  is the energy density,  $p$  is the pressure and  $v$  is the velocity potential. From the action in (2.7), the first order perturbations are governed by

$$E^0_0 : \frac{\kappa^2}{2}\delta\rho = -\frac{k^2}{a^2}\psi(1 + f_T) - 3H(\dot{\psi} + H\phi)(1 + f_T - 12H^2 f_{TT}), \quad (3.4)$$

$$E^i_0 : \frac{\kappa^2}{2}(\rho + p)\partial^i v = -(\partial^i \dot{\psi} + H\partial^i \phi)(1 + f_T) + 12H\dot{H}f_{TT}\partial^i \psi, \quad (3.5)$$

$$E^0_i : \frac{\kappa^2}{2}(\rho + p)\partial_i v = -(\partial_i \dot{\psi} + H\partial_i \phi)(1 + f_T - 12H^2 f_{TT}), \quad (3.6)$$

$$\begin{aligned} E^i_j (i = j) : \\ \frac{\kappa^2}{2}\delta p = -(36H^4\phi + 60H^2\dot{H}\phi + 12H^3\dot{\phi} + 36H^3\dot{\psi} + 36H\dot{H}\dot{\psi} + 12H^2\ddot{\psi})f_{TT} \\ + (3H^2\phi + H\dot{\phi} + 2\dot{H}\phi + 3H\dot{\psi} + \ddot{\psi})(1 + f_T) + 144H^3\dot{H}f_{TTT}(\dot{\psi} + H\phi) \end{aligned} \quad (3.7)$$

$$E^i_j (i \neq j) : \psi - \phi = 0, \quad (3.8)$$

which are the same as those in [12, 18], and we also use  $E^\mu_A$  to denote the equation obtained from variation of the action with respect to  $e_\mu^A$ . Note that  $\partial^i = \delta^{ij}\partial_j$  and  $\partial^2 = \partial^i\partial_i$  throughout this work. Comparing Eq.(3.5) with Eq.(3.6), we approach to an extra scale-independent constraint on  $\phi$ , namely

$$\dot{H}\partial_i \psi = H(\partial_i \dot{\psi} + H\partial_i \phi), \quad (3.9)$$

if  $f_{TT} \neq 0$ . Or equivalently, there are the same number of degrees of freedom as that in GR, but  $f(T)$  theory leads to one more equation. It may lead to inconsistency. Besides, the scale-independent evolution of  $\phi$  in the above equation is incompatible with the integrated Sachs-Wolfe effect.

One may consider  $f_{TT} = 0$ . However, if so,  $f(T) \propto T$  and  $f(T)$  gravity is effectively reduced to GR. In order to solve this puzzle, we go to a more general ansatz in the next subsection.

### 3.2 A general ansatz on perturbed vierbein in $f(T)$ gravity

Since the local Lorentz symmetry is broken down in  $f(T)$  gravity, extra degrees of freedom compared to GR should appear. In this subsection, we include all scalar degrees of freedom in the vierbein. Following [29], the perturbed vierbein is expressed in terms of unperturbed vierbein  $\bar{e}_\mu^A$  and a first-order quantity  $\chi_A^B$  as

$$e_\mu^A = (\delta_B^A + \chi_B^A)\bar{e}_\mu^B, \quad (3.10)$$

with  $\bar{e}_0^A = \delta_0^A$  and  $\bar{e}_i^A = a\delta_i^A$ . In this subsection, we only take into account the scalar degrees of freedom, which are encoded in  $\chi_B^A$  as follows

$$\chi_{AB} = \begin{pmatrix} \phi & \partial_i w \\ \partial_i \tilde{w} & \delta_{ij}\psi + \partial_i \partial_j h + \epsilon_{ijk} \partial^k \tilde{h} \end{pmatrix}. \quad (3.11)$$

Keep in mind that the captical indices are lowered or raised by the Minkowski metric  $\eta_{AB}$  or its inverse. There are six scalar degrees of freedom altogether. The corresponding perturbed metric takes the form:

$$g_{\mu\nu} = \begin{pmatrix} 1 + 2\phi & a\partial_i(w + \tilde{w}) \\ a\partial_i(w + \tilde{w}) & -a^2((1 - 2\psi)\delta_{ij} - 2\partial_i \partial_j h) \end{pmatrix}. \quad (3.12)$$

In general,  $w$  and  $\tilde{w}$  always affect the metric in terms of their combination  $w + \tilde{w}$ , which serves as a single degree of freedom, but as far as the vierbein is concerned, they are two independent degrees of freedom. Moreover,  $\tilde{h}$  does not present itself in the metric as well. To summarize, there are two degrees of freedom which do not contribute to the metric, and this is exactly what we ignore in the previous subsection.

In longitudinal gauge,  $\tilde{w} = -w$  and  $h = 0$ . Since  $w$  has a mass dimension, we introduce a dimensionless quantity  $\zeta$  which is related to  $w$  by  $\zeta = aHw$ . As demonstrated in the appendix, one can obtain the perturbed equations up to first order as follows

$$E^0_0 : \frac{\kappa^2}{2}\delta\rho = a^{-2}(1 + f_T)\partial^2\psi - 12a^{-2}H^2 f_{TT}\partial^2\zeta - 3H(1 + f_T - 12H^2 f_{TT})(\dot{\psi} + H\phi), \quad (3.13)$$

$$E^i_0 : \frac{\kappa^2}{2}(\rho + p)\partial^i v = -(1 + f_T)(\partial^i \dot{\psi} + H\partial^i \phi) + 12H\dot{H}f_{TT}\partial^i \psi, \quad (3.14)$$

$$E^0_i : \frac{\kappa^2}{2}(\rho + p)\partial_i v = -(1 + f_T - 12H^2 f_{TT})(\partial_i \dot{\psi} + H\partial_i \phi) - 4a^{-2}Hf_{TT}\partial_i \partial^2 \zeta, \quad (3.15)$$

$$\text{Tr}(E^i_j) :$$

$$\begin{aligned} \frac{\kappa^2}{2}\delta p = & (1 + f_T) \left( (\ddot{\psi} + 3H\dot{\psi} + H\dot{\phi} + 2\dot{H}\phi + 3H^2\phi) - \frac{1}{3}a^{-2}\partial^2(\psi - \phi) \right) \\ & + f_{TT} \left( -12H^2\ddot{\psi} - 36H(\dot{H} + H^2)\dot{\psi} - 12H^3\dot{\phi} - (60\dot{H}H^2 + 36H^4)\phi \right. \\ & \left. + a^{-2}(8\dot{H} + 4H^2)\partial^2\zeta + 4a^{-2}H\partial^2\dot{\zeta} \right) \\ & + 12f_{TTT}\dot{H}H^2 \left( 12H(\dot{\psi} + H\phi) - 4a^{-2}\partial^2\dot{\zeta} \right), \end{aligned} \quad (3.16)$$

$$E^i_j (i \neq j) : (1 + f_T)\partial_i \partial^j (\phi - \psi) + 12\dot{H}f_{TT}\partial_i \partial^j \zeta = 0. \quad (3.17)$$

See Appendix A in detail. Note that in the above equations, the Parity-violating term  $\tilde{h}$  disappears, but  $w$  survives, even though  $w$  does not appear in the perturbed metric in longitudinal gauge. Compared to the first order equations in GR [28], we have an extra degree of freedom  $\zeta$  in the perturbed equations and one more equation is obtained. Our equations are self-consistent. On the other hand, for the trivial Lagrangian in which  $f(T)$  is a linear function of the torsion scalar  $T$ , all analysis should parallel to those in GR except for a re-scaled coupling constant, and the extra degree of freedom  $\zeta$  should disappear. Here we see that  $\zeta$  always appears in the company of  $f_{TT}$ . It is quite reasonable.

Since the matter Lagrangian is invariant under general coordinate transformation, the energy-momentum tensor should be conserved with respect to the Levi-Civita connection. Then one can find two equations which take exactly the same form as their counterparts in GR:

$$\dot{\delta\rho} + 3H(\delta\rho + \delta p) + a^{-2}(\rho + p)\partial^2 v - 3(\rho + p)\dot{\psi} = 0, \quad (3.18)$$

$$\dot{p}\partial_i v + (\rho + p)\partial_i \dot{v} + \partial_i \delta p + (\rho + p)\partial_i \phi = 0. \quad (3.19)$$

$$\frac{d}{dt}((\rho + p)\partial_i v) + 3H(\rho + p)\partial_i v + \partial_i p + (\rho + p)\partial_i \phi = 0. \quad (3.20)$$

One can also derive the above two equations from Eq.(3.13 -3.17). From now on we will focus on the universe only filled with dust-like matter, namely  $p = \delta p = 0$ , and all equations will be transformed to Fourier space. From Eq.(3.19), one reaches a very useful relation

$$\dot{v} = -\phi. \quad (3.21)$$

Define a gauge invariant fractional matter perturbation

$$\delta_m \equiv \frac{\tilde{\delta\rho}_m}{\rho_m}, \quad (3.22)$$

where

$$\tilde{\delta\rho}_m \equiv \delta\rho_m - 3H\rho_m v \quad (3.23)$$

is the gauge-invariant comoving matter density perturbation. It can also be interpreted as the density perturbation on spacelike hypersurfaces orthogonal to comoving worldlines. Considering Eq.(3.13) and Eq.(3.15), we obtain

$$\frac{\kappa^2}{2}\tilde{\delta\rho}_m = -\frac{k^2}{a^2}\psi(1 + f_T). \quad (3.24)$$

From Eqs. (3.14), (3.17) and (3.22), the evolution of matter density perturbation is given by

$$\dot{\delta}_m = \frac{k^2}{a^2}v - 12H\dot{H}f_{TT}\frac{k^2}{a^2}\frac{\zeta}{\frac{\kappa^2}{2}\rho_m}. \quad (3.25)$$

In order to get the evolution equation of  $\delta_m$ , we need to work out the solution of  $\zeta$  as well.

In this subsection we focus on the non-trivial case with  $f_{TT} \neq 0$ . From Eqs. (3.14) and (3.15), we obtain

$$3\dot{H}\psi = 3H\dot{\psi} + 3H^2\phi + \frac{k^2}{a^2}\zeta. \quad (3.26)$$

On the other hand, Eq. (3.17) can be written by

$$\phi = \psi - 12\dot{H}\frac{f_{TT}}{1 + f_T}\zeta. \quad (3.27)$$

Combing the above two equations, we find

$$\zeta = \frac{3(\dot{H} - H^2)\psi - 3H\dot{\psi}}{\frac{k^2}{a^2} - 36\dot{H}H^2\frac{f_{TT}}{1+f_T}}. \quad (3.28)$$



In the subhorizon limit,  $\zeta \sim \frac{a^2 H^2}{k^2} \psi \ll \psi$ . One can expect that  $\zeta$  will play an important role on the evolution of perturbations at large scales. But here we only focus on the physics in the subhorizon limit and we have

$$\phi \simeq \psi. \quad (3.29)$$

It is the same as that in minimally coupled GR. Therefore, from Eq. (3.14),  $Hv \sim \phi \simeq \psi \gg \zeta$  and hence the second term on the right-hand side of Eq. (3.25) can be neglected. Taking the time derivative of Eq. (3.25), one can obtain

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\phi = 0, \quad (3.30)$$

where Eq.(3.21) is taken into account. Plugging Eq.(3.29) into this equation and combining with Eq.(3.24), the evolution equation of linear matter perturbation becomes

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}}\rho_m\delta_m = 0, \quad (3.31)$$

where  $G_{\text{eff}}$  is the effective Newton's constant which is related to  $G$  by

$$G_{\text{eff}} = \frac{G}{1 + f_T}. \quad (3.32)$$

If  $f(T)$  is a linear function of  $T$ , namely  $f(T) = \alpha T$ , the Newton's constant is just rescaled to be  $G_{\text{eff}} = G/(1 + \alpha)$ , which is also constant in time. We can understand this result from the action (2.7) directly. In  $f(R)$  or scalar-tensor gravity, one can get an analogous equation for fractional matter perturbation  $\delta_m$  with a redefinition of Newton's constant in the short wave-length limit [25].

At the deep matter dominant era, if  $f_T \simeq 0$ , the solution of Eq. (3.31) indicates that the matter density perturbation goes like  $\delta_m \propto a$ . Therefore  $\delta\tilde{\rho}_m = \rho_m\delta_m \sim a^{-2}$  and  $\phi \simeq \psi \sim \text{constant}$  in time. From Eq.(3.26),  $\zeta \sim a^{-1}$  which implies that  $\zeta$  decreases with respect to  $\phi$  and  $\psi$ .

To summarize, we consider all the scalar degrees of freedom in this subsection and obtain the evolution equation of matter energy density perturbation in the subhorizon limit. In this limit, the extra degree of freedom  $\zeta$  is suppressed compared to  $\phi$  and  $\psi$ , but it removes the inconsistency in the former subsection.

### 3.3 Numerical analysis

For convenience, we define the growth as the ratio of the perturbation amplitude at some scale factor relative to some initial scale factor,  $D = \delta_m(a)/\delta_m(a_i)$ . The matter density perturbation  $\delta_m$  is proportional to the scale factor  $a$  in the  $f(T)$  gravity with  $f_T \ll 1$  during the matter era. We introduce a new variable  $g(a)$ , namely

$$g(a) \equiv \frac{D(a)}{a} \quad (3.33)$$

which does not depend on  $a$  during the matter era; thus, the natural choice for the initial conditions are  $g(a_i) = 1, dg/d\ln a|_{a=a_i} = 0$ . From Eq.(3.31), the equation for  $g(a)$  becomes

$$\frac{d^2 g}{d\ln a^2} + \left(4 + \frac{\dot{H}}{H^2}\right) \frac{dg}{d\ln a} + \left(3 + \frac{\dot{H}}{H^2} - \frac{4\pi G_{\text{eff}}\rho_m}{H^2}\right) g = 0. \quad (3.34)$$

This equation reduces to that for dark energy scenario in GR with the same equation of state as the effective dark energy in  $f(T)$  if we replace  $G_{\text{eff}}$  by  $G$ . For a universe only filled with dust-like matter in  $f(T)$  gravity, we have

$$\frac{\dot{H}}{H^2} = -\frac{3}{2} \frac{1 + f/6H^2 + 2f_T}{1 + f_T - 12H^2 f_{TT}}, \quad (3.35)$$

and

$$\frac{4\pi G_{\text{eff}} \rho_m}{H^2} = \frac{3\Omega_m^0}{2(1 + f_T)} \frac{H_0^2}{H^2} a^{-3}, \quad (3.36)$$

here the scale factor  $a_0$  is normalized to be one and  $\Omega_m^0$  is the matter energy density parameter. The superscript ‘0’ denotes that the variables are evaluated at present.

For intance, we consider a power law model with

$$f(T) = \alpha(-T)^n = \alpha(6H^2)^n, \quad (3.37)$$

where  $\alpha$  can be determined by the present Hubble parameter and matter density parameter, namely

$$\alpha = (6H_0^2)^{1-n} (1 - \Omega_m^0) / (2n - 1). \quad (3.38)$$

The equation for the perturbation becomes

$$\begin{aligned} \frac{d^2 g}{d \ln a^2} + \left[ 4 - \frac{3}{2} \frac{1 - h^{2n-2}(1 - \Omega_m^0)}{1 - nh^{2n-2}(1 - \Omega_m^0)} \right] \frac{dg}{d \ln a} \\ + \left[ 3 - \frac{3}{2} \frac{1 - h^{2n-2}(1 - \Omega_m^0)}{1 - nh^{2n-2}(1 - \Omega_m^0)} - \frac{3\Omega_m^0 h^{-2} a^{-3}}{2(1 - \frac{n(1-\Omega_m^0)}{2n-1} h^{2n-2})} \right] g = 0, \end{aligned} \quad (3.39)$$

where  $h \equiv H/H_0$  which is governed by

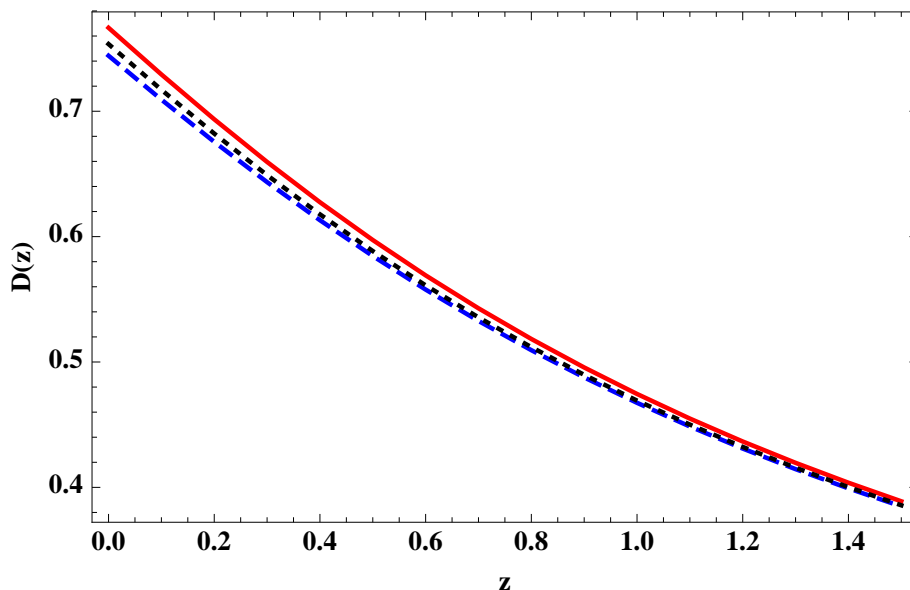
$$\frac{dh^2}{d \ln a} = \frac{-3h^2 + 3h^{2n}(1 - \Omega_m^0)}{1 - nh^{2n-2}(1 - \Omega_m^0)}. \quad (3.40)$$

Since the above equations are too complicated to be solved analytically, we will use numerical method. In [6], Linder pointed out that this model can fit current observation only when  $n \ll 1$ . Therefore we have

$$f_T = \frac{n}{1 - 2n} (1 - \Omega_m^0) \left( \frac{H_0}{H} \right)^{2(1-n)} \quad (3.41)$$

which is much smaller than one during the matter era because  $n \ll 1$  and  $H_0/H \ll 1$ . Here we will adopt  $n = 0.1$  and  $\Omega_m^0 = 0.28$  for numerical calculation. The initial moment should be taken during the matter era, e.g.,  $a_i = 1/31$  (i.e.,  $z = 30$ ). In addition, the initial condition of  $h(a)$  is  $h(a = 1) = 1$ .

When  $f$  is a constant or  $n = 0$ , the term  $f(T)$  acts just as a cosmological constant. In this case, our result recovers that in  $\Lambda$ CDM in GR. For a universe filled with matter and dark energy whose equation of state is the same as that in  $f(T)$  gravity, the evolution equation of  $\delta_m$  in the framework of GR can be obtained by replacing  $G_{\text{eff}}$  with  $G$  in (3.31). Our numerical results are illustrated in Fig. 1. Since  $f_T > 0$ , the effective Newton’s constant in  $f(T)$  gravity gets smaller than that in GR, and the gravitational interaction is weakened. That is why the over-dense perturbation in  $f(T)$  gravity grows slower than that in GR (See the blue dashed and black dotted lines).



**Figure 1:** The numerical solution of the evolution equation for the growth rate of matter perturbation in  $\Lambda$ CDM (the red solid line), power law  $f(T)$  model (the blue dashed line) and dark energy with the same equation of state in GR (the black dotted line).

## 4. Discussions

In this paper we derived the evolution equation for linear matter density perturbation in the framework of  $f(T)$  gravity and compared it to that in GR. We began our analysis from two aspects. One is based on a naive ansatz in which we chose the simplest vierbein and concluded that it leads to an inconsistency in Sec. 3.1. Though different vierbeins are related to each other by a Lorentz transformation, they may have different predictions because the Lorentz symmetry is broken down in  $f(T)$  theory. However, there is not a principle for us to choose a ‘physical’ vierbein. In Sec. 3.2, we proposed a strategy to solve this puzzle. We started with the most general perturbed vierbein and we found that an extra degree of freedom cure the inconsistency in the former case even though it does not appear in the perturbed metric. Finally, in Sec. 3.3, we figured out the growth factor in the power law  $f(T)$  model in detail and showed that the over-dense matter perturbation grows slower than that in GR due to the weakened gravity.

When  $f(T)$  contains some nonlinear terms of  $T$ ,  $f(T)$  gravity is not Lorentz invariant any more. However, the Lorentz symmetry should be preserved at least at small scales. This may require a stringent constraint on  $f(T)$  gravity. Once we take this constraint into account, whether  $f(T)$  theory can lead to an accelerated expansion of our universe is still an open question. We will come back to this question in the future.

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## A. Equations of motion for the perturbations

In the appendix, we present some details of our calculations. From the definition in Eqs. (3.10) and (3.11), one obtains the components of perturbed vierbein  $e_\mu^A$ :

$$\begin{aligned} e_0^0 &= 1 + \phi, & e_i^0 &= a\partial_i\tilde{w}, \\ e_0^i &= -\partial^i w, & e_j^i &= a\left((1-\psi)\delta_j^i - \partial_j\partial^i h - \epsilon_j^{in}\partial_n\tilde{h}\right), \end{aligned}$$

and its inverse  $e^\mu_A$ :

$$\begin{aligned} e^0_0 &= 1 - \phi, & e^i_0 &= a^{-1}\partial^i w, \\ e^0_i &= -\partial_i\tilde{w}, & e^i_j &= a^{-1}\left((1+\psi)\delta_j^i + \partial_j\partial^i h + \epsilon_j^{in}\partial_n\tilde{h}\right). \end{aligned}$$

Plugging the above equations into  $g_{\mu\nu} = \eta_{AB}e_\mu^A e_\nu^B$ , one gets the metric in Eq. (3.12). For simplicity, we choose the Newtonian gauge, where  $\tilde{w} = -w$ , and  $h = 0$ . In this case,

$$\begin{aligned} T^0_{0i} &= -\partial_i\phi - a\partial_i\dot{w}, \\ T^0_{ij} &= 0, \\ T^i_{0j} &= H\delta_j^i - \dot{\psi}\delta_j^i - \epsilon_j^{in}\partial_n\dot{\tilde{h}} + a^{-1}\partial_j\partial^i w, \\ T^i_{jk} &= \partial_k\psi\delta_j^i - \partial_j\psi\delta_k^i + \epsilon_j^{in}\partial_k\partial_n\tilde{h} - \epsilon_k^{in}\partial_j\partial_n\tilde{h}, \end{aligned}$$

and then

$$\begin{aligned} S_0^{0i} &= a^{-2}\partial^i\psi, \\ S_0^{ij} &= -\frac{1}{2}a^{-2}\epsilon^{ijn}\partial_n\dot{\tilde{h}}, \\ S_i^{0j} &= -H\delta_j^i + (\dot{\psi} + 2H\phi - \frac{1}{2}a^{-1}\partial^2 w)\delta_j^i + \frac{1}{2}a^{-1}\partial_i\partial^j w, \\ S_i^{jk} &= -\frac{1}{2}a^{-2}\epsilon^{jkn}\partial_i\partial_n\tilde{h} + \frac{1}{2}a^{-2}(\partial^k\psi - \partial^k\phi - a\partial^k\dot{w})\delta_i^j - \frac{1}{2}a^{-2}(\partial^j\psi - \partial^j\phi - a\partial^j\dot{w})\delta_i^k. \end{aligned}$$

In addition, one can also calculate perturbed torsion scalar, namely

$$T = -6H^2 + 12H(\dot{\psi} + H\phi) - 4a^{-1}H\partial^2 w. \quad (\text{A.1})$$

Defining  $\tilde{T}^\alpha_A = e^\rho_A \mathbf{T}_\rho^\alpha$ , one can easily obtain

$$\tilde{T}^0_0 = -\rho - \delta\rho + \rho\phi, \quad (\text{A.2})$$

$$\tilde{T}^0_i = a^{-1}(\rho + p)\partial_i v - \rho\partial_i w, \quad (\text{A.3})$$

$$\tilde{T}^i_0 = -a^{-2}(\rho + p)\partial^i v + a^{-1}p\partial^i w, \quad (\text{A.4})$$

$$\tilde{T}^i_j = a^{-1}(p + \delta p)\delta_j^i + a^{-1}p\psi\delta_j^i + a^{-1}p\epsilon_j^{in}\partial_n\tilde{h}. \quad (\text{A.5})$$

Plugging these expressions into the equation of motion Eq. (2.8), we find  $E^0_0$ :

$$\frac{\kappa^2}{2}\rho = \frac{1}{4}(T + f) + 3H^2(1 + f_T), \quad (\text{A.6})$$

and

$$\begin{aligned} \frac{\kappa^2}{2}(-\delta\rho + \rho\phi) &= \frac{1}{4}(T + f)\phi + (1 + f_T)(3H\dot{\psi} + 6H^2\phi - a^{-2}\partial^2\psi) \\ &\quad - 3H^2f_{TT}(12H(\dot{\psi} + H\phi) - 4a^{-1}H\partial^2w). \end{aligned} \quad (\text{A.7})$$

Note that Eq. (A.6) describes evolution of homogeneous background and Eq.(A.7) equation of motion for density perturbation  $\delta\rho$ . Combing Eq. (A.6) and Eq. (A.7) yields Eq. (3.13).

$E^0_i$ :

$$\begin{aligned} \frac{\kappa^2}{2}(a^{-1}(\rho + p)\partial_i v - \rho\partial_i w) &= -\frac{1}{4}(T + f)\partial_i w + (1 + f_T)(-a^{-1}\partial_i\dot{\psi} - a^{-1}H\partial_i\phi - 3H^2\partial_i w) \\ &\quad + f_{TT}(12a^{-1}H^2(\partial_i\dot{\psi} + H\partial_i\phi) - 4a^{-2}H^2\partial_i\partial^2w). \end{aligned} \quad (\text{A.8})$$

Considering Eq. (A.6), one can obtain Eq. (3.14).

$E^i_0$ :

$$\begin{aligned} \frac{\kappa^2}{2}(-a^{-2}(\rho + p)\partial^i v + a^{-1}p\partial^i w) &= -\frac{1}{4}a^{-1}(T + f)\partial^i w - 12a^{-2}H\dot{H}f_{TT}(\partial^i\psi - aH\partial^i w) \\ &\quad + (1 + f_T)\left(a^{-2}\partial^i(\dot{\psi} + H\phi) - a^{-1}(\dot{H} + 3H^2)\partial^i w\right) \end{aligned} \quad (\text{A.9})$$

$E^i_j$ :

$$\frac{\kappa^2}{2}p = -\frac{1}{4}(T + f) - (\dot{H} + 3H^2)(1 + f_T) + 12\dot{H}H^2f_{TT}, \quad (\text{A.10})$$

$$\begin{aligned} &\frac{\kappa^2}{2}(\delta p\delta_j^i + p\psi\delta_j^i + p\epsilon_j^{in}\partial_n\tilde{h}) \\ &= -\frac{1}{4}(T + f)(\psi\delta_j^i + \epsilon_j^{in}\partial_n\tilde{h}) + (1 + f_T)\left((\ddot{\psi} + 3H\dot{\psi} - \dot{H}\psi - 3H^2\psi + H\dot{\phi} + 2\dot{H}\phi + 3H^2\phi)\delta_j^i \right. \\ &\quad \left. - (\dot{H} + 3H^2)\epsilon_j^{in}\partial_n\tilde{h} - \frac{1}{2}a^{-2}\partial^2(\psi - \phi)\delta_j^i + \frac{1}{2}a^{-2}\partial_j\partial^i(\psi - \phi)\right) \\ &\quad + f_{TT}\left((-12H^2\ddot{\psi} - 36H(\dot{H} + H^2)\dot{\psi} + 12\dot{H}H^2\psi - 12H^3\dot{\phi} - (60\dot{H}H^2 + 36H^4)\phi)\delta_j^i \right. \\ &\quad \left. + a^{-1}(14\dot{H}H\partial^2w + 8H^3\partial^2w + 4H^2\partial^2\psi)\delta_j^i + 12\dot{H}H^2\epsilon_j^{in}\partial_n\tilde{h} - 6a^{-1}\dot{H}H\partial_j\partial^iw\right) \\ &\quad + 12f_{TTT}\dot{H}H^2\left(12H(\dot{\psi} + H\phi) - 4a^{-1}H\partial^2w\right)\delta_j^i. \end{aligned} \quad (\text{A.11})$$

Plugging Eq. (A.10) into (A.9), one reaches Eq.(3.15). Using Eq. (A.10), one can simplify Eq.(A.11) as

$$\begin{aligned}
& \frac{\kappa^2}{2} \delta p \delta_j^i \\
&= (1 + f_T) \left( (\ddot{\psi} + 3H\dot{\psi} + H\dot{\phi} + 2\dot{H}\phi + 3H^2\phi) \delta_j^i - \frac{1}{2} a^{-2} \partial^2 (\psi - \phi) \delta_j^i + \frac{1}{2} a^{-2} \partial_j \partial^i (\psi - \phi) \right) \\
&+ f_{TT} \left( (-12H^2 \ddot{\psi} - 36H(\dot{H} + H^2) \dot{\psi} - 12H^3 \dot{\phi} - (60\dot{H}H^2 + 36H^4) \phi) \delta_j^i \right. \\
&\quad \left. + a^{-1} (14\dot{H}H\partial^2 w + 8H^3 \partial^2 w + 4H^2 \partial^2 \dot{w}) \delta_j^i - 6a^{-1} \dot{H}H \partial_j \partial^i w \right) \\
&+ 12f_{TTT} \dot{H}H^2 \left( 12H(\dot{\psi} + H\phi) - 4a^{-1} H \partial^2 w \right) \delta_j^i. \tag{A.12}
\end{aligned}$$

Taking the trace of Eq.(A.12), one obtains

$$\begin{aligned}
\frac{\kappa^2}{2} \delta p &= (1 + f_T) \left( (\ddot{\psi} + 3H\dot{\psi} + H\dot{\phi} + 2\dot{H}\phi + 3H^2\phi) - \frac{1}{3} a^{-2} \partial^2 (\psi - \phi) \right) \\
&+ f_{TT} \left( (-12H^2 \ddot{\psi} - 36H(\dot{H} + H^2) \dot{\psi} - 12H^3 \dot{\phi} - (60\dot{H}H^2 + 36H^4) \phi) \right. \\
&\quad \left. + a^{-1} (12\dot{H}H\partial^2 w + 8H^3 \partial^2 w + 4H^2 \partial^2 \dot{w}) \right) \\
&+ 12f_{TTT} \dot{H}H^2 \left( 12H(\dot{\psi} + H\phi) - 4a^{-1} H \partial^2 w \right). \tag{A.13}
\end{aligned}$$

Noticing that  $\zeta = aHw$ , we can rewrite this equation in terms of Eq.(3.16). In addition, by combining Eq.(A.12) and Eq.(A.13), one can obtain Eq.(3.17) after expressing  $w$  in terms of the dimensionless quantity  $\zeta$ . Here we want to stress that the terms with  $\tilde{h}$  in (A.11) are cancelled and thus  $\tilde{h}$  does not show up in the perturbation equations in Sec. 3.2.

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